2D Grushin-type equations: minimal time and null controllable data

Karine Beauchard, Luc Miller, Morgan Morancey

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K. Beauchard [*], L. Miller [†], M. Morancey [‡§]

Abstract

We study internal null controllability for degenerate parabolic equations of Grushin-type $G_\gamma = \partial_{xx}^2 + |x|^{2\gamma}\partial_{yy}^2$, $(\gamma > 0)$, in the rectangle $(x, y) \in \Omega = (-1, 1) \times (0, 1)$.

Previous works proved that null controllability holds for weak degeneracies (γ small), and fails for strong degeneracies (γ large). Moreover, in the transition regime and with strip shaped control domains, a positive minimal time is required.

In this paper, we work with controls acting on two strips, symmetric with respect to the degeneracy. We give the explicit value of the minimal time and we characterize the initial data that can be steered to zero in time T (when the system is not null controllable): their regularity depends on the control domain and the time T.

We also prove that, with a control that acts on one strip, touching the degeneracy line $\{x = 0\}$, then Grushin-type equations are null controllable in any time $T > 0$ and for any degeneracy $\gamma > 0$.

Our approach is based on a precise study of the observability property for the one-dimensional heat equations satisfied by the Fourier coefficients in variable y. This precise study is done, through a transmutation process, on the resulting one-dimensional wave equations, by lateral propagation of energy method.

Key words: Grushin operator, null controllability, degenerate parabolic equations, minimal time, geometric control condition, Carleman estimates, transmutation.

AMS subject classifications: 35K65, 93B05, 93B07

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Contents

1 Introduction

1.1 Main results

We consider Grushin-type equations

$$\begin{cases} \partial_t f - \partial_{xx}^2 f - |x|^{2\gamma}\partial_{yy}^2 f = u(t,x,y)\mathbf{1}_\omega(x,y), & (t,x,y)\in(0,T)\times\Omega, \\ f(t,x,y)=0, & (t,x,y)\in(0,T)\times\partial\Omega, \\ f(0,x,y)=f^0(x,y), & (x,y)\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega := (-1,1)\times(0,1)$, $\gamma>0$ and $\mathbf{1}_\omega$ denotes the characteristic function of the subset ω. It is a degenerate parabolic equation, since the coefficient of $\partial_{yy}^2 f$ vanishes on the line $\{x=0\}$. System (1.1) is a linear control system in which the state is f and the control is the locally distributed source term u. We are interested in its null controllability, in the following sense.

Definition 1.1 (Null controllability)**.** Let $T>0$ and $\omega\subset\Omega$. System (1.1) is null controllable from ω in time T if, for every $f^0\in L^2(\Omega,\mathbb{R})$, there exists $u\in L^2((0,T)\times\Omega,\mathbb{R})$ such that the associated solution of (1.1) satisfies $f(T,.,.)=0$.

System (1.1) is null controllable from ω if there exists $T>0$ such that system (1.1) is null controllable from ω in time T.

In [6], Beauchard, Cannarsa and Guglielmi proved the following result.

Theorem 1.1. *Let ω be an open subset of $(-1,1)\times(0,1)$ such that $\overline{\omega}\subset(0,1)\times(0,1)$.*

1. *If $\gamma\in(0,1)$, then system (1.1) is null controllable from ω in any time $T>0$.*

2. *If $\gamma=1$ and $\omega=(a,b)\times(0,1)$ where $0<a<b\leqslant 1$, then a positive minimal time is required for null controllability from ω; moreover*

$$T_{min} := \inf\{T>0; \textit{system (1.1) is null controllable from } \omega \textit{ in time } T\}, \tag{1.2}$$

   satisfies $T_{min}\geqslant \frac{a^2}{2}$.

3. *If $\gamma>1$, then system (1.1) is not null controllable from ω.*

In particular, null controllability holds for weak degeneracies ($0<\gamma<1$), fails for strong degeneracies ($\gamma>1$) and, in the transition regime ($\gamma=1$), a positive minimal time is required.

The goal of the present article is to go further in this direction, and to give

- the explicit value of the minimal time T_{min},
- a characterization of the initial conditions that can be steered to zero, when the system is not null controllable.

Our first result is the null controllability in any positive time (null minimal time) and with any degeneracy $\gamma>0$, when the control acts on a strip touching the degeneracy line $\{x=0\}$.

Theorem 1.2. *Let $b\in(0,1)$ and $\gamma\geqslant 1$. System (1.1) is null controllable from $(0,b)\times(0,1)$ in any time $T>0$.*

Remark 1.1. The null controllability from $(0,b)\times(0,1)$ in any time was already known for $\gamma\in(0,1)$ from Theorem 1.1.

Our second result is the computation of the minimal time, when the control acts on two strips symmetric with respect to the degeneracy. Let

$$\omega_{a,b} := (-b,-a) \cup (a,b) \tag{1.3}$$

Theorem 1.3. *Let $0 < a < b \leqslant 1$, $\omega := \omega_{a,b} \times (0,1)$ and $\gamma = 1$. The minimal time required for the null controllability of system (1.1) from ω is $T_{min} = \frac{a^2}{2}$. More precisely,*

i) for every $T > \frac{a^2}{2}$, system (1.1) is null controllable from ω in time T,

ii) for every $T \leqslant \frac{a^2}{2}$, system (1.1) is not null controllable from ω in time T.

The quantity $\frac{a^2}{2}$ appears in our proof as the Agmon distance, associated to potential $q(x) = x^2$, between $\omega_{a,b}$ (related to the control support) and $\{0\}$ (related to the degeneracy location). We recall that, for a given potential $q \in C^1(\mathbb{R}, \mathbb{R}^+)$, the associated Agmon distance between two points $z, c \in \mathbb{R}$, with $z < c$ is

$$d_{Ag}(z,c) := \int_z^c \sqrt{q(s)}ds.$$

When system (1.1) is not null controllable, we characterize the initial data that can be steered to zero by L^2-controls

$$\begin{aligned}\mathcal{F}_T := \{f^0 \in L^2(\Omega);\quad & \exists u \in L^2((0,T)\times\Omega,\mathbb{R}) \text{ such that} \\ & \text{the solution of (1.1) satisfies } f(T,.,.) = 0\}.\end{aligned}$$

Their analyticity regularity takes part in this characterization: elements of $\mathcal{F}_T$ need to be sufficiently analytic in variable y.

Definition 1.2 (Space $\mathcal{A}_\alpha$)**.** For $\alpha \in \mathbb{R}^+$, $\mathcal{A}_\alpha$ is the space of functions that are analytic in variable y with values in $L^2((-1,1),\mathbb{R})$ in variable x, defined by

$$\mathcal{A}_\alpha := \left\{ f(x,y) = \sum_{n\in\mathbb{N}^*} f_n(x)\sin(n\pi y); \sum_{n\in\mathbb{N}^*} e^{2\alpha n}\|f_n\|^2_{L^2(-1,1)} < \infty \right\}.$$

Theorem 1.4. *Let $0 < a < 1$, $\omega := \omega_{a,1} \times (0,1)$, $\gamma \geqslant 1$ and $T > 0$. We assume that either $[\gamma > 1]$ or $[\gamma = 1$ and $T < \frac{a^2}{2}]$. Then*

$$\inf\{\alpha \in \mathbb{R}^+; \mathcal{A}_\alpha \subset \mathcal{F}_T\} = \begin{cases} \pi \frac{a^{\gamma+1}}{\gamma+1}, & \text{if } \gamma > 1, \\ \pi\left(\frac{a^2}{2} - T\right), & \text{if } \gamma = 1 \text{ and } T < \frac{a^2}{2}. \end{cases}$$

The quantity $\frac{a^{\gamma+1}}{\gamma+1}$ appears in our proof as the Agmon distance, associated to potential $q(x) = |x|^{2\gamma}$, between $\omega_{a,1}$ and $\{0\}$.

Theorem 1.4 emphasizes an influence of the control domain ω and the time T, on the set of null controllable initial conditions $\mathcal{F}_T$. It is then natural to state the following conjecture: for $\gamma > 0$, $\omega \subset \Omega$ and $T > 0$ given, the regularity of the initial conditions f^0 that are null controllable in time T for system (1.1) depends on (γ, ω, T). But this characterization remains an open problem, for an arbitrary control support ω.

1.2 Sketch of the proof, structure of the article

The proof of the above results rely on

- Hilbert Uniqueness Method, which proves the equivalence between the null controllability of the 2D system (1.1) and the observability of its 2D adjoint system,
- a Fourier expansion of the solution of (1.1) (or its adjoint system) in variable y,
$$f(t,x,y)=\sum_{n\in\mathbb{N}^*} f_n(t,x)\sin(n\pi y).$$

These tools are explained in Section 2.

The proof of the positive controllability results (Theorems 1.2 and 1.3 *i)*), uses the equivalence between the observability of the 2D adjoint system and the observability of the 1D heat equations solved by the Fourier modes, uniformly with respect to the Fourier frequency $n \in \mathbb{N}^*$.

For Theorem 1.2, this uniform 1D observability is proved thanks to a global Carleman estimate. Observing until $\{x=0\}$ gives more latitude in the construction of the weight functions than in the proof of Theorem 1.1 in [6]. This is the key point to conclude for any $\gamma>0$. The proof of Theorem 1.2 is performed in Section 3.

For Theorem 1.3*i)*, we use transmutation method from 1D heat equations to 1D wave equations. By lateral propagation of energy method on the resulting 1D wave equations, we get observability constants which are exponential in the Fourier frequency, in an optimal way. These observability constants are sharper in the present article than the one proved in [6] by Carleman estimates. These constants can be compensated by dissipation of the 1D heat equations if $T>\frac{a^2}{2}$. The proof of Theorem 1.3 is performed in Section 4.

The proof of Theorem 1.4 also takes advantage of the optimality of the 1D observability constants obtained above. The exponential dependance of the 1D observability constant is compensated by the analyticity regularity of the initial data. The proof of Theorem 1.4 is performed in Section 5.

1.3 Comments and conjectures

The presence of a second strip in the control region is related to the techniques used (the 1D wave equation propagates in both directions). If the control acts only on one strip $(a,b)\times(0,1)$, then the proof developped in the present article would lead to null controllability of (1.1) from $(a,b)\times(0,1)$ in any time $T>\frac{1+a^2}{2}$. However, we conjecture that the minimal time is still $\frac{a^2}{2}$ in this configuration.

When the degeneracy function x^2 in Grushin's operator is replaced by a potential $q(x)$, which vanishes at a non degenerate minimum and the two control strips are at the same Agmon distance defined by q from this minimum, then *the critical control time is this Agmon distance.*

More precisely, when

- the degeneracy function x^2 is replaced by an absolutely continuous function $q(x)$ which is twice differentiable at 0 and satisfies $q''(0) > 0$, $q(x) > 0 = q(0)$ for $x \neq 0$,
- the strips of the control set $\omega = [(-1,-b) \cup (a,1)] \times (0,1)$ are at the same Agmon distance $d_q(\omega) = \int_0^a \sqrt{q(s)}ds = \int_{-b}^0 \sqrt{q(s)}ds$ from the degeneracy set $\{x = 0\}$

then null-controllability from ω in time T of $\partial_t - \partial_x^2 - q(x)\partial_y^2$ holds for $T > \frac{d_q(\omega)}{\theta}$ and does not hold for $T < \frac{d_q(\omega)}{\theta}$, where $\theta = \sqrt{\frac{q''(0)}{2}}$.

The positive result may be proved by a refinement of Haraux's sideways energy method and the negative result may be proved by Agmon estimates, as in Allibert's paper [2] for the boundary control of the wave equation on the cylindrical surface of a barrel.

1.4 Bibliographical comments

1.4.1 Null controllability of the heat equation

The null controllability of the heat equation is a well understood subject. In particular, the heat equation on a smooth bounded domain Ω of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$), with a source term located on an open subset ω of Ω is null controllable in arbitrary time $T > 0$ and with an arbitrary control support ω. This result is due, for the case $d = 1$, to Fattorini and Russell [19], and, for $d \geqslant 2$, to Fursikov and Imanuvilov [21] and Lebeau and Robbiano [25]. The literature contains many further developments.

1.4.2 Boundary-degenerate parabolic equations

The null controllability of 1D parabolic equations degenerating on the boundary of the space domain is well-understood. In particular, null controllability still holds for weak degeneracies and fails for too strong ones, see [12, 13, 11, 1, 29, 10, 9, 8, 20, 34]. Fewer results are available for multidimensional problems, mainly in the case of two dimensional parabolic operators which simply degenerate in the normal direction to the boundary of the space domain, see [14].

1.4.3 Parabolic equations degenerating inside the domain

In [28], the authors study linearized Crocco type equations

$$\begin{cases} \partial_t f + \partial_x f - \partial_{vv}^2 f = u(t,x,v)1_\omega(x,v), & (t,x,v) \in (0,T) \times \mathbb{T} \times (0,1), \\ f(t,x,0) = f(t,x,1) = 0, & (t,x) \in (0,T) \times \mathbb{T}. \end{cases}$$

For a given strict open subset ω of $\mathbb{T} \times (0,1)$, they prove that null controllability does not hold: the optimal result is regional null controllability.

Then, the literature contains results about Grushin-type equations recalled in Theorems 1.1 (see [6]). Approximate controllability for a Grushin operator with singular coefficients is obtained by Morancey in [31].

Similar results are known for Kolmogorov-type equations

$$\begin{cases} \partial_t f - \partial_{vv}^2 f + v^\gamma \partial_x f = u(t,x,v)\mathbf{1}_\omega(x,v), & (t,x,v) \in (0,T) \times \Omega^{(K)}, \\ f(t,x,\pm 1) = 0, & (t,x) \in (0,T) \times \mathbb{T}, \end{cases}$$

with $\gamma \in \mathbb{N}^*$ and $\Omega^{(K)} := \mathbb{T} \times (-1,1)$, $\mathbb{T}$ being the 1D torus. (see [5, 7]).

1.4.4 Other occurences of minimal times in a parabolic context

Even though it does not seem natural at first sight to have a minimal time for controllability of parabolic equations this phenomena has already been enlightened in previous works such as [15] by Dolecki in the case of a pointwise control. Recently, for null controllability of systems of parabolic equations, the existence and the characterization of the minimal time has been obtained by Ammar Khodja, Benabdallah, Gonzalez-Burgos, and De Teresa [3, 4].

1.4.5 Transmutation method

The proof developed in the present article uses the transmutation strategy and precise estimates on the resulting wave equations with respect to the potential. Due to technical difficulties, we do not use the transmutation strategy from the point of view of controllability as in [32, 30] by Phung and Miller but from the observability point of view as done in [18] by Ervedoza and Zuazua. This strategy has already been used by Gueye [22] for boundary control of a boundary degenerate operator in divergence form.

The estimates on the resulting wave equations rely on a lateral (or 'sideways') propagation of energy (that is where the role of the time and space variables are exchanged) as used by Haraux [23] and Zuazua [35] or more recently by Haraux, Liard and Privat [24]. The use of both a transmutation method and lateral propagation of energy was already suggested in [17] by Duyckaerts, Zhang and Zuazua.

1.4.6 Agmon distance, analytic control

In [26], Lebeau study the boundary control of the wave equation, when unique continuation holds, but the geometric control condition does not hold. He proves a quantification of the analyticity regularity required for the initial conditions to be null controllable.

In [2], Allibert study the same question over surfaces of revolution. His strategy relies on Fourier series expansion (with respect to the angle variable), as in the present paper. Moreover, his work emphasizes the role Agmon distance plays in the observabilty constants for wave equations. However, his context is quite different from ours. Indeed, Allibert focuses on boundary control for wave equations, whereas we focus on interior control of Grushin-type equations. As a consequence, our intermediary results on wave equations are different from Allibert's.

2 Well posedness, Fourier decomposition, dissipation speed

In this section, we recall well-posedness results for the Grushin equation (1.1) in Section 2.1 and Hilbert Uniqueness Method in Section 2.2. In Section 2.3, we justify the expansion in Fourier series of the solutions of the 2D system and we reduce the 2D observability problem to the observability of 1D heat equations, uniformly with respect to the Fourier parameter. Finally, in Section 2.4, we recall results concerning the decay rate of these 1D heat equations.

2.1 Well posedness

Here, we recall well posedness result for (1.1) proved in [6, Section 2]. For $f, g \in C_0^\infty(\Omega)$, we introduce

$$(f,g) := \int_\Omega \Big(f_x g_x + |x|^{2\gamma} f_y g_y\Big) \mathrm{d}x\mathrm{d}y.$$

Let $|\cdot|_V := (\cdot,\cdot)^{1/2}$ and $V := \overline{C_0^\infty(\Omega)}^{|\cdot|_V}$, which is a dense subspace of $L^2(\Omega)$. The operator G_γ, defined by

$$\begin{gathered} D(G_\gamma) := \left\{ f \in V \,;\, \exists c > 0 \text{ such that } |(f,h)| \leq c\|h\|_{L^2},\, \forall h \in V \right\}, \\ \langle G_\gamma f, h\rangle := -(f,h),\, \forall h \in V, \end{gathered}$$

is self-adjoint on $L^2(\Omega)$, generates an analytic semigroup and satisfies

$$G_\gamma f = \partial^2_{xx} f + |x|^{2\gamma} \partial^2_{yy} f, \quad \text{a.e. on } \Omega.$$

This implies the following result.

Proposition 2.1. *For any $\gamma > 0$, $f^0 \in L^2(\Omega)$, $T > 0$ and $u \in L^2((0,T), L^2(\Omega))$ there exists a unique solution f in $C^0([0,T], L^2(\Omega)) \cap L^2((0,T), V)$ of problem*

$$\begin{cases} f'(t) = G_\gamma f(t) + u(t), & t \in [0,T], \\ f(0) = f^0. \end{cases}$$

It satisfies $f(t) \in D(G_\gamma)$ and $f'(t) \in L^2(\Omega)$ for a.e. $t \in (0,T)$ and

$$\|f(t)\|_{L^2} \leq \|f^0\|_{L^2} + \sqrt{T}\|u\|_{L^2((0,T)\times\Omega)}, \quad \forall t \in [0,T].$$

2.2 Null controllability and observability

By duality, we consider the adjoint system of (1.1)

$$\begin{cases} \partial_t g - \partial_{xx}^2 g - |x|^{2\gamma} \partial_{yy}^2 g = 0, & (t,x,y) \in (0,T) \times \Omega, \\ g(t,x,y) = 0, & (t,x,y) \in (0,T) \times \partial\Omega, \\ g(0,x,y) = g^0(x,y), & (x,y) \in \Omega. \end{cases} \tag{2.1}$$

Definition 2.1 (Observability)**.** Let $T > 0$. System (2.1) is observable from ω in time T if there exists $C > 0$ such that, for every $g^0 \in L^2(\Omega)$, the solution of (2.1) satisfies

$$\int_\Omega g(T,x,y)^2 \mathrm{d}x\mathrm{d}y \leqslant C \int_0^T \int_\omega g(t,x,y)^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t.$$

System (2.1) is observable from ω if there exists $T > 0$ such that it is observable from ω in time T.

By Hilbert uniqueness method, system (1.1) is null controllable from ω in time T if and only if system (2.1) is observable from ω in time T. In this article, our strategy relies on this equivalence.

2.3 Fourier expansion and uniform observability

Let us consider the solution of (1.1). Since f belongs to $C([0,T], L^2(\Omega))$, the function $y \mapsto f(t,x,y)$ belongs to $L^2(0,1)$ for a.e. $(t,x) \in (0,T) \times (-1,1)$, thus it can be developed in Fourier series with respect to y as follows

$$f(t,x,y) = \sum_{n \in \mathbb{N}^*} f_n(t,x) \varphi_n(y), \tag{2.2}$$

where

$$\varphi_n(y) := \sqrt{2} \sin(n\pi y), \quad \forall n \in \mathbb{N}^*,$$

and

$$f_n(t,x) := \int_0^1 f(t,x,y) \varphi_n(y) \mathrm{d}y, \quad \forall n \in \mathbb{N}^*. \tag{2.3}$$

We also develop $u \in L^2((0,T) \times \Omega)$ in Fourier series in the variable y as

$$u(t,x,y) = \sum_{n \in \mathbb{N}^*} u_n(t,x) \varphi_n(y).$$

We consider strip shaped control domains $\omega := \omega_x \times (0,1)$. The following result follows from [6, Proposition 2]

Proposition 2.2. *For every $n \geq 1$, f_n is the unique weak solution of*

$$\begin{cases} \partial_t f_n - \partial_{xx}^2 f_n + (n\pi)^2 |x|^{2\gamma} f_n = u_n(t,x) \mathbf{1}_{\omega_x}(x), & (t,x) \in (0,T) \times (-1,1), \\ f_n(t, \pm 1) = 0, & t \in (0,T), \\ f_n(0,x) = f_n^0(x), & x \in (-1,1), \end{cases} \tag{2.4}$$

where $f_n^0 \in L^2(-1,1)$ *is given by* $f_n^0(x) := \int_0^1 f^0(x,y)\varphi_n(y)dy$*.*

Using Parseval identity, it comes that (2.1) is observable from $\omega = \omega_x \times (0,1)$ in time T if and only if the adjoint system of (2.4)

$$\begin{cases} \partial_t g_n - \partial_{xx}^2 g_n + (n\pi)^2 |x|^{2\gamma} g_n = 0, & (t,x) \in (0,T) \times (-1,1), \\ g_n(t,\pm 1) = 0, & t \in (0,T), \\ g_n(0,x) = g_n^0(x), & x \in (-1,1), \end{cases} \tag{2.5}$$

is uniformly observable from ω_x in time T in the following sense.

Definition 2.2 (Cost of observability)**.** Let $T > 0$, $\omega_x \subset (-1,1)$ and $d \geqslant 0$. System (2.5) is observable from ω_x in time T with exponential cost d if there exists $C = C(d) > 0$ such that, for every $n \in \mathbb{N}^*$ and $g_n^0 \in L^2(-1,1)$, the solution of (2.5) satisfies

$$\left(\int_{-1}^1 g_n(T,x)^2 \mathrm{d}x \right)^{1/2} \leqslant C e^{dn} \left(\int_0^T \int_{\omega_x} g_n(t,x)^2 \mathrm{d}x \mathrm{d}t \right)^{1/2}.$$

System (2.5) is uniformly observable from ω_x in time T if it is observable from ω_x in time T with exponential cost $d = 0$.

In all what follows, null controllability of 2D systems will be studied through uniform observability of the associated sequence of 1D problems.

The following proposition links the null controllability of analytic initial conditions of (1.1) and the cost of observability (when not uniform) of (2.5).

Proposition 2.3. *Let* $\gamma \geqslant 1$*,* $T > 0$ *and* ω_x *be an open subset of* $(0,1)$*. Assume that*

$$d^* := \inf\{d > 0; \quad \textit{system (2.5) is observable from } \omega_x \textit{ in time } T \textit{ with exponential cost } d\} \tag{2.6}$$

is positive. Then $\inf\{\alpha > 0; \mathcal{A}_\alpha \subset \mathcal{F}_T\} = d^*$*.*

Proof of Proposition 2.3. Using a classical duality argument (see [16, 27] by Dolecki, Russell and Lions for pioneer works and [33] by Tucsnak and Weiss for a complete overview), if (2.5) is observable from ω_x in time T with exponential cost α, then the linear map

$$\mathcal{U}_{T,n} : f_n^0 \in L^2((-1,1),\mathbb{R}) \mapsto u_n \in L^2((0,T) \times \omega_x, \mathbb{R}),$$

where u_n is the control of minimal L^2 norm steering the solution of (2.4) from f_n^0 to 0, is well defined and its norm is the smallest observability constant for (2.5). In particular

$$\|\mathcal{U}_{T,n}\|_{\mathcal{L}[L^2(-1,1),L^2((0,T)\times(-1,1))]} \leqslant C(\alpha) e^{n\alpha},$$

where $C(\alpha)$ is as in Definition 2.2.

Step 1: We prove that $\mathcal{A}_\alpha \subset \mathcal{F}_T$ for every $\alpha > d^*$. Let $\alpha > d^*$ and $f^0 \in \mathcal{A}_\alpha$, i.e.

$$\sum_{n\in\mathbb{N}^*} e^{2\alpha n}\|f_n^0\|^2_{L^2(-1,1)} < \infty.$$

Let $u(t,x,y) := \sum_{n\in\mathbb{N}^*} u_n(t,x)\varphi_n(y)$ where $u_n := \mathcal{U}_{T,n}(f_n^0)$. Then u is supported on $\omega_x \times (0,1)$, steers the solution of (1.1) from f^0 to 0 and belongs to $L^2((0,T)\times\Omega,\mathbb{R})$ because

$$\begin{aligned}\int_0^T\int_\Omega u(t,x,y)^2 dxdydt &= \sum_{n\in\mathbb{N}^*}\int_0^T\int_{-1}^1 u_n(t,x)^2 dxdt \\ &\leqslant \sum_{n\in\mathbb{N}^*} C(\alpha)^2 e^{2\alpha n}\|f_n^0\|^2_{L^2(-1,1)} < \infty.\end{aligned}$$

Step 2: We prove that, for every $\alpha < d^*$, $\mathcal{A}_\alpha$ is not contained in $\mathcal{F}_T$. Let $\alpha \in (0,d^*)$. By assumption, there exists an extraction $(n_k)_{k\in\mathbb{N}^*}$ such that $\|\mathcal{U}_{T,n_k}\| > ke^{\alpha n_k}$ for every $k\in\mathbb{N}^*$. Thus, there exists $\widetilde{f}^0_{n_k} \in L^2((-1,1),\mathbb{R})$ such that $\|\widetilde{f}^0_{n_k}\|_{L^2} = 1$ and

$$\left\|\mathcal{U}_{T,n_k}\left(\widetilde{f}^0_{n_k}\right)\right\|_{L^2((0,T)\times(-1,1))} \geqslant ke^{\alpha n_k}.$$

Let $f^0 \in \mathcal{A}_\alpha$ be defined by

$$f^0(x,y) := \sum_{k\in\mathbb{N}^*}\widetilde{f}^0_{n_k}(x)\frac{e^{-\alpha n_k}}{k}\varphi_{n_k}(y).$$

Assume that there exists $u \in L^2((0,T)\times\Omega,\mathbb{R})$ that steers the solution of (1.1) from f^0 to 0. Then, $u_{n_k}(t,x) := \int_0^1 u(t,x,y)\varphi_{n_k}(y)\mathrm{d}y$ steers the solution of (2.4) from $\widetilde{f}^0_{n_k}\frac{e^{-\alpha n_k}}{k}$ to 0. Thus,

$$\begin{aligned}\|u_{n_k}\|_{L^2((0,T)\times(-1,1))} &\geqslant \left\|\mathcal{U}_{T,n_k}\left(\widetilde{f}^0_{n_k}\frac{e^{-\alpha n_k}}{k}\right)\right\|_{L^2((0,T)\times(-1,1))} \\ &\geqslant ke^{\alpha n_k}\frac{e^{-\alpha n_k}}{k} = 1.\end{aligned}$$

This is in contradiction with the fact that u belongs to $L^2((0,T)\times\Omega,\mathbb{R})$. $\square$

2.4 Dissipation speed

Let us introduce, for every $n\in\mathbb{N}^*, \gamma > 0$, the operator $G_{n,\gamma}$ defined on $L^2(-1,1)$ by

$$D(G_{n,\gamma}) := H^2\cap H^1_0(-1,1), \quad G_{n,\gamma}\varphi := -\varphi'' + (n\pi)^2|x|^{2\gamma}\varphi. \tag{2.7}$$

The smallest eigenvalue of $G_{n,\gamma}$ is given by

$$\lambda_{n,\gamma} = \min\left\{\frac{\int_{-1}^1\left[v'(x)^2 + (n\pi)^2|x|^{2\gamma}v(x)^2\right]\mathrm{d}x}{\int_{-1}^1 v(x)^2\mathrm{d}x}; v\in H^1_0(-1,1),\ v\neq 0\right\}. \tag{2.8}$$

We are interested in the asymptotic behavior (as $n \to +\infty$) of $\lambda_{n,\gamma}$, which quantifies the dissipation speed of the solution of (2.5). The following result is proved in [6, Lemma 2, Lemma 4 and Proposition 4].

Proposition 2.4. *For every $\gamma > 0$, there are constants $c_*, c^* > 0$ such that*

$$c_* n^{\frac{2}{1+\gamma}} \leqslant \lambda_{n,\gamma} \leqslant c^* n^{\frac{2}{1+\gamma}}, \quad \forall n \in \mathbb{N}^*.$$

There exists a unique positive solution with $L^2(-1,1)$ norm one of problem

$$\begin{cases} -v_{n,\gamma}''(x) + (n\pi)^2|x|^{2\gamma}v_{n,\gamma}(x) = \lambda_{n,\gamma}v_{n,\gamma}(x), \quad x \in (-1,1), \\ v_{n,\gamma}(\pm 1) = 0. \end{cases}$$

This solution $v_{n,\gamma}$ is even. Moreover, there exists $C > 0$ such that

$$n\pi \leqslant \lambda_{n,1} \leqslant n\pi + C, \qquad \forall n \in \mathbb{N}^*. \tag{2.9}$$

3 Observability on a strip touching the degeneracy line

The goal of this section is the proof of Theorem 1.2. In Sect. 3.1, we reduce the problem to that of controllability from $(0,1) \times (0,1)$. Then, it suffices to prove observability from $(0,1) \times (0,1)$ which relies on a global Carleman estimate for solutions of (2.5), stated in Sect. 3.2 and proved in Appendix.

3.1 Towards the study of controllability from the half domain

The following proposition states that it is sufficient to consider the case $b = 1$.

Proposition 3.1. *Let $\gamma \geqslant 1$, $b \in (0,1)$ and $T > 0$. If system (1.1) with $\Omega = (-1,b) \times (0,1)$ is null controllable from $\omega = (0,b) \times (0,1)$ in time T then system (1.1) with $\Omega = (-1,1) \times (0,1)$ is null controllable from $\omega = (0,b) \times (0,1)$ in time T.*

The property given in assumption can be proved with the strategy developed in this article : the fact that the interval $x \in (-1,b)$ is non-symmetric plays no role; the only important thing is that we control up to the boundary.

Proof of Proposition 3.1. The proof relies on a cut-off argument. Let $\delta \in (0, \frac{b}{2})$, $\chi_1 \in C^\infty([-1,1],[0,1])$ such that

$$\chi_1 = 1, \text{ on } (b-\delta, 1),$$
$$\chi_1 = 0, \text{ on } (-1, \delta)$$

and $\chi_2 := 1 - \chi_1$. We introduce the domains

$$\Omega_1 := (b-\delta, 1) \times (0,1), \qquad \omega_1 := (b-\delta, b) \times (0,1),$$
$$\Omega_2 := (-1, b-\delta) \times (0,1), \qquad \omega_2 := (0, b-\delta) \times (0,1).$$

Let $f^0 \in L^2(\Omega)$ and $T > 0$. By uniform parabolicity in the case $j = 1$ (see for example [21]) and by assumption and rescalling in the case $j = 2$, there exist $u_j \in L^2((0,T) \times \Omega_j)$ such that the solutions of

$$\begin{cases} \partial_t f_j - \partial_{xx}^2 f_j - |x|^{2\gamma} \partial_{yy}^2 f_j = u_j(t,x,y) \mathbf{1}_{\omega_j}(x,y), & (t,x,y) \in (0,T) \times \Omega_j, \\ f_j(t,x,y) = 0, & (t,x,y) \in (0,T) \times \partial\Omega_j, \\ f_j(0,x,y) = f^0(x,y), & (x,y) \in \Omega_j, \end{cases}$$

satisfy $f_j(T,.,.) = 0$ on Ω_j for $j = 1,2$. Then $f(t,x,y) := \sum_{j=1,2} \chi_j(x) f_j(t,x,y)$ solves (1.1) with

$$u(t,x,y) := \sum_{j=1,2} \Big(u_j(t,x,y) \mathbf{1}_{\omega_j}(x,y) - \chi_j''(x) f_j(t,x,y) - 2\chi_j'(x) \partial_x f_j(t,x,y) \Big)$$

which belongs to $L^2((0,T) \times \Omega)$ by Proposition 2.1 and is supported on $(0,T) \times (0,b) \times (0,1)$.

□

3.2 A global Carleman estimate

For $n \in \mathbb{N}^*$ and $\gamma > 0$, we introduce the operator

$$\mathcal{P}_{n,\gamma} g := \partial_t g - \partial_{xx}^2 g + (n\pi)^2 |x|^{2\gamma} g.$$

Proposition 3.2. *Let $\gamma \geqslant 1$. There exist a weight function $\beta \in C^1([-1,1],[1,+\infty))$ and a positive constant $\mathcal{C}_1$ such that for every $T > 0$, $M \geqslant M_* = M_*(T,\beta)$, $n \in \mathbb{N}^*$ and $g \in C^0([0,T], L^2(-1,1)) \cap L^2((0,T), H_0^1(-1,1))$ the following inequality holds*

$$\mathcal{C}_1 \int_0^T \int_{-1}^1 \frac{M^3}{(t(T-t))^3} \big| g(t,x) \big|^2 e^{-\frac{M\beta(x)}{t(T-t)}} \mathrm{d}x\mathrm{d}t \leqslant \int_0^T \int_{-1}^1 |\mathcal{P}_{n,\gamma} g|^2 e^{-\frac{M\beta(x)}{t(T-t)}} \mathrm{d}x\mathrm{d}t$$
$$+ \int_0^T \int_0^1 \left(\frac{M^3}{(t(T-t))^3} + \frac{Mn^2}{t(T-t)} \right) |g(t,x)|^2 e^{-\frac{M\beta(x)}{t(T-t)}} \mathrm{d}x\mathrm{d}t. \tag{3.1}$$

We refer to the Appendix for a complete proof, which follows the usual strategy. Note that M does not need to depend on n, contrarily to what happens in reference [6], where the observability region ω does not touch the degeneracy line $\{x = 0\}$. Our weight β is inspired by the classical one (see (7.1), (7.2), (7.3) and (7.4)), but it is an appropriate adaptation to our situation (see (7.5), (7.6)).

3.3 Uniform observability

The Carleman estimate of Proposition 3.2 allows to prove the following result, which implies Theorem 1.2.

Proposition 3.3. *Let* $\gamma \in [1, +\infty)$ *and* $T > 0$. *System (2.5) is uniformly observable from* $(0, 1)$ *in time* T.

Proof of Proposition 3.3. For $t \in (T/3, 2T/3)$, we have

$$\frac{4}{T^2} \leqslant \frac{1}{t(T-t)} \leqslant \frac{9}{2T^2}$$

and

$$\int_{-1}^{1} g(T,x)^2 \mathrm{d}x \leqslant \int_{-1}^{1} g(t,x)^2 \mathrm{d}x e^{-\lambda_{n,\gamma}\frac{T}{3}}.$$

Thus, by Proposition 3.2.

$$\mathcal{C}_1 \frac{64M^3}{T^6} e^{-\frac{9M\beta^*}{2T^2}} \frac{T}{3} e^{\lambda_{n,\gamma}\frac{T}{3}} \int_{-1}^{1} g(T,x)^2 \mathrm{d}x \leqslant \mathcal{C}_3(1+n^2) \int_0^T \int_0^1 g(t,x)^2 \mathrm{d}x\mathrm{d}t$$

where $\beta^* := \max\{\beta(x); x \in [-1,1]\}$, $\beta_* := \min\{\beta(x); x \in [-1,1]\}$ and $\mathcal{C}_3 := \max\{(x+x^3)e^{-\beta_* x}; x \geqslant 0\}$. By Proposition 2.4, we get

$$\int_{-1}^{1} g(T,x)^2 \mathrm{d}x \leqslant \mathcal{C}_4 T^5 e^{c_1 \frac{M}{T^2} - c_2 n^{\frac{2}{1+\gamma}} T}(1+n^2) \int_0^T \int_0^1 g(t,x)^2 \mathrm{d}x\mathrm{d}t \tag{3.2}$$

for some constants $c_1, c_2, \mathcal{C}_4 > 0$ (independent of n, T and g). This gives the conclusion.

□

4 Minimal time

The goal of this section is the proof of Theorem 1.3 through the study of uniform observability of (2.5). Thus, in the whole section, we take $\gamma = 1$.

In Section 4.1, through transmutation method, we associate 1D wave equations to the 1D heat equations (2.5). We then study an observability-like estimate for these wave equations. In Sections 4.2, using a lateral propagation of energy method, we give a precise estimate of the exponential cost appearing in this observability-like inequality. We prove Theorem 1.3 *i)* in Section 4.3 getting back to the 2D equation and using the dissipation of Fourier modes. Finally, we prove Theorem 1.3 *ii)* in Section 4.4.

4.1 Strategy and transmutation method

We state in Proposition 4.1 that it is sufficient to study null controllability from $\omega_{a,1} \times (0, 1)$.

Proposition 4.1. *Assume that $0 < a < b \leqslant 1$ and $T > 0$. If system (1.1) is null controllable in time $T > 0$ from $\omega_{a,1} \times (0,1)$ then system (1.1) is null controllable in time T from $\omega_{a,b} \times (0,1)$.*

The proof of this proposition is very similar to that of Proposition 3.1 (except that there is here two strips in the control domain) and is left to the reader. Then, Theorem 1.3 *i)* is a consequence of the following result.

Proposition 4.2. *Let $a \in (0,1)$ and $T > \frac{a^2}{2}$. System (2.5) is uniformly observable on $\omega_{a,1}$ in time T.*

This observability property will be studied on a 1D wave equation associated through transmutation. We here recall the properties needed for the transmutation strategy. For simplicity of notations, the partial derivatives are sometimes denoted by subscripts i.e. $g_x = \partial_x g$.

Using an integral transformation, the transmutation strategy induces observability properties of the heat equation (2.5) from the study of the following wave equation

$$\begin{cases} w_{ss} - w_{xx} + (n\pi)^2 x^2 w = 0, & (s,x) \in (-L,L) \times (-1,1), \\ w(s,\pm 1) = 0, & s \in (-L,L), \\ (w,w_s)(0,x) = (w_0,w_1)(x), & x \in (-1,1). \end{cases} \tag{4.1}$$

The integral transformation is based on the following kernel which existence is proved in [18, Proposition 3.1].

Proposition 4.3. *Let $T > 0$ and $L > 0$. For any $\nu > 3L^2$, there exists a unique solution of*

$$\begin{cases} \partial_t k_T(t,s) + \partial_{ss}^2 k_T(t,s) = 0, & (t,s) \in (0,T) \times (-L,L), \\ k_T(0,s) = k_T(T,s) = 0, & s \in (-L,L), \\ k_T(t,0) = 0, & t \in (0,T), \\ \partial_s k_T(t,0) = exp\left(-\frac{\nu T}{t(T-t)}\right), & t \in (0,T). \end{cases} \tag{4.2}$$

Moreover, for any $(t,s) \in (0,T) \times (-L,L)$, this solution satisfies

$$|k_T(t,s)| \leqslant |s| exp\left(\frac{2}{\min(t,T-t)}\left(s^2 - \frac{\nu}{3}\right)\right),$$
$$|\partial_s k_T(t,s)| \leqslant exp\left(\frac{2}{\min(t,T-t)}\left(s^2 - \frac{\nu}{3}\right)\right).$$

Once this kernel is defined, the transmutation from a solution of the parabolic equation (2.5) to a solution of the hyperbolic equation (4.1) is given by the following proposition (see [18, Theorem 2.1] for the proof).

Proposition 4.4. *Let $g^0 \in L^2(-1,1)$ and g be the associated solution of (2.5). We define*

$$w(s,x) := \int_0^T k_T(t,s) g(t,x) \mathrm{d}t, \quad (s,x) \in (-L,L) \times (-1,1). \tag{4.3}$$

Then, w is the unique solution of (4.1) associated to the initial conditions

$$w_0(x) := 0, \quad w_1(x) := \int_0^T exp\left(-\frac{\nu T}{t(T-t)}\right) g(t,x)\mathrm{d}t. \tag{4.4}$$

4.2 Study of wave equations

For the sake of generality, we consider an abstract potential $q \in C^1([-1,1],\mathbb{R})$ i.e. we study the system

$$\begin{cases} w_{ss} - w_{xx} + q(x)w = 0, & (s,x) \in (-L,L) \times (-1,1), \\ w(s,\pm 1) = 0, & s \in (-L,L), \\ (w,w_s)(0,x) = (w_0,w_1)(x), & x \in (-1,1). \end{cases} \tag{4.5}$$

This subsection is dedicated to the proof of precise estimates of the solution of (4.5) with respect to the potential q.

Proposition 4.5. *Let $L > 1$ and $\delta \in (0,a)$ be such that $a + 3\delta < 1$. There exists $C > 0$ such that, for every $\tilde{\delta} > 0$, $q \in C^1([-1,1],\mathbb{R}^+)\backslash\{0\}$, $(w_0,w_1) \in H_0^1 \times L^2(-1,1)$, the associated solution of (4.5) satisfies*

$$\|w_0\|_{H_0^1}^2 + \|w_1\|_{L^2}^2 \leqslant C O_W(a,q,\tilde{\delta},\delta) \int_{-L}^{L} \int_{\omega_{a,1}} \left(w_s^2 + w^2\right)(s,x)\mathrm{d}x\mathrm{d}s, \tag{4.6}$$

where

$$O_W(a,q,\tilde{\delta},\delta) = \max\left(e^{\int_0^{a+2\delta}[M(y)+2\sqrt{\tilde{q}(y)}]\mathrm{d}y}, e^{\int_{-a-2\delta}^{0}[M(y)+2\sqrt{\tilde{q}(y)}]\mathrm{d}y}\right), \tag{4.7}$$

$$\tilde{q} : x \in (-1,1) \mapsto q(x) + \tilde{\delta}^2 ||q||_{L^\infty(-1,1)},$$
$$M : x \in (-1,1) \mapsto \frac{|\tilde{q}'(x)|}{\tilde{q}(x)}.$$

Remark 4.1. Note that the Agmon distance associated to the potential $\tilde{q}$ appears in the observability-like constant $O_W(a,q,\tilde{\delta},\delta)$.

Because of the term in w, the inequality (4.6) is not a classical observability inequality. However using the definition of w in (4.3) we will be able, in the next subsection, to relate (4.6) to the observability of (2.5).

Proof of Proposition 4.5. Let $\varepsilon \in (0, L-1)$ be such that $L > a+2\delta+\varepsilon$. We use lateral propagation of the energy, inspired by Haraux [23, Proposition 1.4]. It is sufficient to prove (4.6) for q, w_0, w_1 in $C_c^\infty((-1,1),\mathbb{R})$. We will use a modified energy defined, by

$$F(x) := \int_{-x-\varepsilon}^{x+\varepsilon} \left(w_s^2(s,x) + w_x^2(s,x) + \tilde{q}(x)w^2(s,x)\right)\mathrm{d}s, \quad \forall x \in (0,1). \tag{4.8}$$

First step : Growth of F.
Differentiating F, integrating by parts and using equation (4.5) lead to

$$\begin{aligned}F'(x) &= \big(w_s^2 + w_x^2 + \tilde{q}(x)w^2\big)(x+\varepsilon, x) + \big(w_s^2 + w_x^2 + \tilde{q}(x)w^2\big)(-x-\varepsilon, x)\\ &\quad + \int_{-x-\varepsilon}^{x+\varepsilon} \tilde{q}'(x)w^2(s,x)\mathrm{d}s + 2\int_{-x-\varepsilon}^{x+\varepsilon} \big(w_s w_{sx} + w_x w_{xx} + \tilde{q}(x)ww_x\big)(s,x)\mathrm{d}s\\ &\geqslant \tilde{q}(x)w^2(x+\varepsilon,x) + \tilde{q}(x)w^2(-x-\varepsilon,x) + \int_{-x-\varepsilon}^{x+\varepsilon} \tilde{q}'(x)w^2(s,x)\mathrm{d}s\\ &\quad + 2\int_{-x-\varepsilon}^{x+\varepsilon} \big(q(x)ww_x + \tilde{q}(x)ww_x\big)(s,x)\mathrm{d}s.\end{aligned}$$

Then, as $q(x) \leqslant \tilde{q}(x)$, and using the definition of M gives

$$\begin{aligned}F'(x) &\geqslant -M(x)\int_{-x-\varepsilon}^{x+\varepsilon} \tilde{q}(x)w^2(s,x)\mathrm{d}s - 2\sqrt{\tilde{q}(x)}\int_{-x-\varepsilon}^{x+\varepsilon} \big(\tilde{q}(x)w^2 + w_x^2\big)(s,x)\mathrm{d}s\\ &\geqslant -\big(M(x) + 2\sqrt{\tilde{q}(x)}\big)F(x).\end{aligned}$$

Hence, the map

$$x \in (0,1) \mapsto F(x)\exp\left(\int_0^x M(y) + 2\sqrt{\tilde{q}(y)}\mathrm{d}y\right) \tag{4.9}$$

is non-decreasing.

Second step : Estimates on $(0, a+\delta)$.
From (4.9), it comes that for any $j \in \{0, \dots, \lfloor \frac{a}{\delta} \rfloor\}$, for any $x \in (j\delta, (j+1)\delta)$,

$$F(x) \leqslant \exp\left(\int_x^{x+a-(j-1)\delta} [M(y) + 2\sqrt{\tilde{q}(y)}]\mathrm{d}y\right) F(x + a - (j-1)\delta).$$

Then integrating for $x \in (j\delta, (j+1)\delta)$ leads to

$$\int_{j\delta}^{(j+1)\delta} F(x)\mathrm{d}x \leqslant \exp\left(\int_0^{a+2\delta} [M(y) + 2\sqrt{\tilde{q}(y)}]\mathrm{d}y\right)\int_{a+\delta}^{a+2\delta} F(x)\mathrm{d}x.$$

Using the same strategy for $x \in (\lfloor \frac{a}{\delta} \rfloor\delta, a+\delta)$ and summing the inequalities then leads to

$$\int_0^{a+\delta} F(x)\mathrm{d}x \leqslant \lceil \tfrac{a}{\delta} \rceil \exp\left(\int_0^{a+2\delta} [M(y) + 2\sqrt{\tilde{q}(y)}]\mathrm{d}y\right)\int_{a+\delta}^{a+2\delta} F(x)\mathrm{d}x. \tag{4.10}$$

We now estimate the term in w_x in the right-hand side of (4.10). Let $\zeta \in C_c^\infty((-L,L)\times(a,a+3\delta),[0,1])$ be such that

$$\zeta \equiv 1 \text{ on } [-a-2\delta-\varepsilon, a+2\delta+\varepsilon] \times [a+\delta, a+2\delta].$$

Multiplying the equation (4.5) by w an using the definition of ζ it comes that

$$\int_{-a-2\delta-\varepsilon}^{a+2\delta+\varepsilon}\int_{a+\delta}^{a+2\delta} w_x^2(s,x)\mathrm{d}x\mathrm{d}s \leqslant \int_{-L}^{L}\int_{a}^{a+3\delta} \zeta(s,x)w_x^2(s,x)\mathrm{d}x\mathrm{d}s$$
$$\leqslant \int_{-L}^{L}\int_{a}^{a+3\delta} \zeta(s,x)\big(w_s^2 - q(x)w^2 + (ww_x)_x - (ww_s)_s\big)(s,x)\mathrm{d}x\mathrm{d}s.$$

After integration by parts and using $q \geqslant 0$, we obtain

$$\int_{-a-2\delta-\varepsilon}^{a+2\delta+\varepsilon}\int_{a+\delta}^{a+2\delta} w_x^2(s,x)\mathrm{d}x\mathrm{d}s \leqslant C(\zeta)\int_{-L}^{L}\int_{a}^{a+3\delta} \big(w_s^2 + w^2\big)(s,x)\mathrm{d}x\mathrm{d}s.$$

Then, going back to (4.10) and using again $q \leqslant \tilde{q}$ gives

$$\int_{-\varepsilon}^{\varepsilon}\int_{0}^{a+\delta} \big(w_s^2 + w_x^2 + q(x)w^2\big)(s,x)\mathrm{d}x\mathrm{d}s \leqslant \int_{0}^{a+\delta} F(x)\mathrm{d}x$$
$$\leqslant Ce^{\int_0^{a+2\delta}[M(y)+2\sqrt{\tilde{q}(y)}]\mathrm{d}y}\int_{-L}^{L}\int_{a}^{a+3\delta} \big(w_s^2 + w^2\big)(s,x)\mathrm{d}x\mathrm{d}s. \tag{4.11}$$

Third step : estimates on $(a+\delta, 1)$.
Let $\zeta_2 \in C_c^\infty((-L,L)\times(a,2),[0,1])$ be such that

$$\zeta_2 \equiv 1 \text{ on } [-\varepsilon,\varepsilon]\times[a+\delta,1],$$

The same arguments as previously lead to

$$\int_{-\varepsilon}^{\varepsilon}\int_{a+\delta}^{1} w_x^2(s,x)\mathrm{d}x\mathrm{d}s \leqslant C(\zeta_2)\int_{-L}^{L}\int_{a}^{1} \big(w_s^2 + w^2\big)(s,x)\mathrm{d}x\mathrm{d}s.$$

Together with (4.11) this leads to

$$\int_{-\varepsilon}^{\varepsilon}\int_{0}^{1} \big(w_s^2 + w_x^2 + q(x)w^2\big)(s,x)\mathrm{d}x\mathrm{d}s$$
$$\leqslant Ce^{\int_0^{a+2\delta} M(y)+2\sqrt{\tilde{q}(y)}\mathrm{d}y}\int_{-L}^{L}\int_{a}^{1} \big(w_s^2 + w^2\big)(s,x)\mathrm{d}x\mathrm{d}s. \tag{4.12}$$

Fourth step : conclusion.
The exact same strategy on $(-1,0)$ leads to

$$\int_{-\varepsilon}^{\varepsilon}\int_{-1}^{0} \big(w_s^2 + w_x^2 + q(x)w^2\big)(s,x)\mathrm{d}x\mathrm{d}s$$
$$\leqslant Ce^{\int_{-a-2\delta}^{0}[M(y)+2\sqrt{\tilde{q}(y)}]\mathrm{d}y}\int_{-L}^{L}\int_{-1}^{-a} \big(w_s^2 + w^2\big)(s,x)\mathrm{d}x\mathrm{d}s. \tag{4.13}$$

Defining the classical energy

$$E(s) := \int_{-1}^{1} \big(w_s^2 + w_x^2 + q(x)w^2\big)(s,x)\mathrm{d}x \tag{4.14}$$

and using its conservation we get

$$2\varepsilon E(0) = \int_{-\varepsilon}^{\varepsilon}\int_{-1}^{1} \big(w_s^2 + w_x^2 + q(x)w^2\big)(s,x)\mathrm{d}x\mathrm{d}s.$$

Then, summing (4.12) and (4.13) and using the non-negativity of q ends the proof of Proposition 4.5.

$\square$

4.3 Upper bound on the minimal time

This subsection is dedicated to the proof of Proposition 4.2 which implies Theorem 1.3 *i)*.

Proof of Proposition 4.2. Let $a \in (0,1)$ and $T > \frac{a^2}{2}$. Let L and δ satisfying the assumptions of Proposition 4.5. Up to a reduction of δ, we set $\eta, \tilde{\delta} > 0$ such that

$$T > \frac{(a+2\delta)^2}{2} + \tilde{\delta}(a+2\delta) + \eta. \tag{4.15}$$

Let $n \in \mathbb{N}^*$ and $g_n^0 \in L^2(-1,1)$. Let g_n be the associated solution of (2.5). For the sake of brevity, we omit the subscript n in the rest of the proof. We consider k_η the transmutation kernel associated to the time η defined in Proposition 4.3, and w the associated solution of (4.1) given by Proposition 4.4.

Recall that according to Proposition 4.4,

$$w(s,x) = \int_0^\eta k_\eta(t,s)g(t,x)\mathrm{d}t, \quad (s,x) \in (-L,L)\times(-1,1),$$
$$w_0(x) = 0, \quad x \in (-1,1),$$
$$w_1(x) = \int_0^\eta exp\left(-\frac{\nu\eta}{t(\eta-t)}\right) g(t,x)\mathrm{d}t, \quad x \in (-1,1),$$

where $\nu > 3L^2$.

Then, Proposition 4.5 applied to the case $q(x) := (n\pi)^2x^2$, implies the existence of $C > 0$ independent of n and g^0 such that

$$\|w_1\|_{L^2}^2 \leqslant CO_W(a,q,\delta,\tilde{\delta}) \int_{-L}^{L}\int_{\omega_{a,1}} \big(w_s^2 + w^2\big)(s,x)\mathrm{d}x\mathrm{d}s. \tag{4.16}$$

In all what follows, C denotes a positive constant that may vary but which is always independent of n and g^0.

Using the definitions of M and $\tilde{q}$ given in Proposition 4.5, it comes that M does not depend on n and thus we get

$$\begin{aligned} O_W(a,q,\delta,\tilde{\delta}) &= \exp\left(\int_0^{a+2\delta} [M(y)+2\sqrt{\tilde{q}(y)}]\mathrm{d}y\right) \\ &\leqslant C\exp\left(2n\pi\left(\frac{(a+2\delta)^2}{2}+\tilde{\delta}(a+2\delta)\right)\right). \end{aligned} \tag{4.17}$$

We now relate the left and right-hand sides of (4.16) to the function g. Straightforward computations imply

$$\begin{aligned} \int_{-L}^{L}\int_{\omega_{a,1}} w^2(s,x)\mathrm{d}x\mathrm{d}s &= \int_{-L}^{L}\int_{\omega_{a,1}} \left(\int_0^{\eta} k_\eta(t,s)g(t,x)\mathrm{d}t\right)^2 \mathrm{d}x\mathrm{d}s \\ &\leqslant \|k_\eta\|^2_{L^2((0,\eta)\times(-L,L))}\int_0^{\eta}\int_{\omega_{a,1}} g^2(t,x)\mathrm{d}x\mathrm{d}t \\ &\leqslant \|k_\eta\|^2_{L^2((0,\eta)\times(-L,L))}\int_0^{T}\int_{\omega_{a,1}} g^2(t,x)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.18}$$

In the same way, it comes that

$$\begin{aligned} \int_{-L}^{L}\int_{\omega_{a,1}} w_s^2(s,x)\mathrm{d}x\mathrm{d}s &= \int_{-L}^{L}\int_{\omega_{a,1}} \left(\int_0^{\eta} \partial_s k(t,s)g(t,x)\mathrm{d}t\right)^2 \mathrm{d}x\mathrm{d}s \\ &\leqslant \|\partial_s k_\eta\|^2_{L^2((0,\eta)\times(-L,L))}\int_0^{T}\int_{\omega_{a,1}} g^2(t,x)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.19}$$

Thus, using (4.18) and (4.19) in (4.16) leads to

$$\|w_1\|^2_{L^2} \leqslant Ce^{2n\pi\left(\frac{(a+2\delta)^2}{2}+\tilde{\delta}(a+2\delta)\right)}(\|k_\eta\|^2_{L^2}+\|\partial_s k_\eta\|^2_{L^2})\int_0^T\int_{\omega_{a,1}} g^2(t,x)\mathrm{d}x\mathrm{d}t. \tag{4.20}$$

To give a lower estimate on the left side of (4.16) we use a spectral decomposition. For a fixed $n \in \mathbb{N}^*$, we denote by $(\lambda_j^{(n)})_{j\in\mathbb{N}^*}$ and $(\varphi_j^{(n)})_{j\in\mathbb{N}^*}$ the nondecreasing sequence of eigenvalues and the associated sequence of L^2 normalized eigenvectors of the operator $-\partial^2_{xx}+(n\pi)^2x^2$ with domain $H^2\cap H^1_0(-1,1)$ i.e.

$$\begin{cases} \left(-\partial^2_{xx}+(n\pi)^2x^2\right)\varphi_j^{(n)} = \lambda_j^{(n)}\varphi_j^{(n)}, \\ \varphi_j^{(n)}(-1) = \varphi_j^{(n)}(1) = 0. \end{cases}$$

For any $g^0 \in L^2(-1,1)$, expanded in the basis $(\varphi_j^{(n)})_{j\in\mathbb{N}^*}$ as

$$g^0 = \sum_{j=1}^{+\infty} g_j^0\varphi_j^{(n)},$$

the associated solution g of (2.5) reads as

$$g(t) = \sum_{j=1}^{+\infty} e^{-\lambda_j^{(n)} t} g_j^0 \varphi_j^{(n)}. \tag{4.21}$$

Then, straightforward computations lead to

$$\begin{aligned}
\|w_1\|^2_{L^2(-1,1)} &= \left\| \int_0^\eta exp\left(-\frac{\nu\eta}{t(\eta-t)}\right) g(t,\cdot)\mathrm{d}t \right\|^2_{L^2(-1,1)} \\
&= \sum_{j=1}^{+\infty} |g_j^0|^2 \left| \int_0^\eta exp\left(-\frac{\nu\eta}{t(\eta-t)}\right) e^{-\lambda_j^{(n)} t}\mathrm{d}t \right|^2 \\
&\geqslant \left| \int_0^\eta exp\left(-\frac{\nu\eta}{t(\eta-t)}\right) \mathrm{d}t \right|^2 \sum_{j=1}^{+\infty} |g_j^0|^2 e^{-2\lambda_j^{(n)}\eta} \\
&= \left| \int_0^\eta exp\left(-\frac{\nu\eta}{t(\eta-t)}\right) \mathrm{d}t \right|^2 \|g(\eta)\|^2_{L^2(-1,1)}.
\end{aligned} \tag{4.22}$$

Finally, using (4.20) and (4.22) in the observability inequality (4.16) leads to the existence of $C > 0$ such that

$$\|g(\eta)\|^2_{L^2(-1,1)} \leqslant C e^{2n\pi\left(\frac{(a+2\delta)^2}{2} + \tilde{\delta}(a+2\delta)\right)} \int_0^T \int_{\omega_{a,1}} g^2(t,x)\mathrm{d}x\mathrm{d}t. \tag{4.23}$$

We conclude using the dissipation speed given by Proposition 2.4

$$\lambda_1^{(n)} \geqslant \pi n, \quad \forall n \in \mathbb{N}^*. \tag{4.24}$$

Thus,

$$\|g(T)\|^2_{L^2(-1,1)} \leqslant e^{-2n\pi(T-\eta)} \|g(\eta)\|^2_{L^2(-1,1)}. \tag{4.25}$$

Together with (4.23) we get the existence of $C > 0$ such that

$$\|g(T)\|^2_{L^2(-1,1)} \leqslant C e^{2n\pi\left(\frac{(a+2\delta)^2}{2} + \tilde{\delta}(a+2\delta) + \eta - T\right)} \int_0^T \int_{\omega_{a,1}} g^2(t,x)\mathrm{d}x\mathrm{d}t. \tag{4.26}$$

Thus, the choice of constants $\delta, \tilde{\delta}, \eta$ in (4.15) implies the uniform observability of (2.5) and Proposition 4.2.

□

Remark 4.2. We see in (4.19) that the term in w in the right-hand side of the observability-like inequality is easily dealt with.

From (4.17) and (4.25) we see that the key point in the proof of Theorem 1.3 is the competition between the cost of observability of the wave equation (4.1) and the dissipation of the heat equation (2.5). Both are of the same order of magnitude when $\gamma = 1$ and thus the dissipation compensates the cost of observability only for $T > \frac{a^2}{2}$.

Remark 4.3. Notice that, by Theorem 1.3 *ii)*, the exponential cost in (4.17) is optimal with respect to the power of n and with respect to the multiplicative constant in front of this power of n.

4.4 Lower bound on the minimal time

The goal of this section is the proof of Theorem 1.3 *ii)*, which is a consequence of the following result

Proposition 4.6. *Let* $0 < a < b \leqslant 1$ *and* $T \leqslant \frac{a^2}{2}$. *System (2.5) is not uniformly observable on* $\omega_{a,b}$ *in time* T.

Proof of Proposition 4.6. This proof follows the one of [6, Lemma 4]. Let $\gamma = 1$ and $T \leqslant \frac{a^2}{2}$. We design a sequence of solutions of (2.5) such that

$$\frac{\int_0^T \int_{\omega_{a,b}} g_n(t,x)^2 \mathrm{d}x\mathrm{d}t}{\int_{-1}^1 g_n(T,x)^2 \mathrm{d}x} \underset{n\to+\infty}{\longrightarrow} 0. \tag{4.27}$$

Let $v_{n,1}$ be the eigenfunction associated to the smallest eigenvalue of $G_{n,1}$ in Proposition 2.4. For any $n \in \mathbb{N}^*$, let

$$g_n(t,x) := e^{-\lambda_{n,1}t} v_{n,1}(x), \quad \forall (t,x) \in (0,+\infty) \times (-1,1). \tag{4.28}$$

Then g_n solves (2.5) and

$$\int_{-1}^1 g_n(T,x)^2 \mathrm{d}x = e^{-2\lambda_{n,1}T},$$
$$\int_0^T \int_{\omega_{a,b}} g_n(t,x)^2 \mathrm{d}x\mathrm{d}t = \frac{1-e^{-2\lambda_{n,1}T}}{2\lambda_{n,1}} \int_{\omega_{a,b}} v_{n,1}(x)^2 \mathrm{d}x.$$

As $v_{n,1}$ is even, to get (4.27) it suffices to prove that

$$\frac{e^{2\lambda_{n,1}T}}{\lambda_{n,1}} \int_a^b v_{n,1}^2(x)\mathrm{d}x \underset{n\to\infty}{\longrightarrow} 0. \tag{4.29}$$

From [6, Lemma 4], it comes that

$$\int_a^b v_{n,1}^2(x)\mathrm{d}x \underset{n\to\infty}{\sim} \frac{e^{-a^2 n\pi}}{2a\pi\sqrt{n}}. \tag{4.30}$$

By (2.9), this implies (4.29). $\square$

5 Null controllable initial conditions

The goal of this section is the proof of Theorem 1.4. The observability-like inequality for the wave equation proved in the previous section also enables us to characterize null controllable initial conditions. By Proposition 2.3, Theorem 1.4 is equivalent to the following result.

Proposition 5.1. *Let $0 < a < 1$, $\gamma \geqslant 1$, $T > 0$ and $\omega_x := (-1,-a) \cup (a,1)$. We assume that either $[\gamma > 1]$ or $[\gamma = 1$ and $T < \frac{a^2}{2}]$. Then, the explicit value of the quantity d^* defined in (2.6) is*

$$d^* = \begin{cases} \pi \frac{a^{\gamma+1}}{\gamma+1}, & \text{if } \gamma > 1, \\ \pi \left(\frac{a^2}{2} - T \right), & \text{if } \gamma = 1 \text{ and } T < \frac{a^2}{2}. \end{cases}$$

Proof of Proposition 5.1. Step 1: We prove Proposition 5.1 for $\gamma = 1$ and $T < \frac{a^2}{2}$.

Step 1.1: We prove that $d^* \leqslant \pi \left(\frac{a^2}{2} - T \right)$. Let $d > \pi \left(\frac{a^2}{2} - T \right)$. We want to prove that system (2.5) is observable from ω_x in time T with exponential cost d. It is a consequence of (4.26), up to a reduction of $\delta, \tilde{\delta}, \eta$.

Step 1.2: We prove that $d^* \geqslant \pi \left(\frac{a^2}{2} - T \right)$. Let $d < \pi \left(\frac{a^2}{2} - T \right)$. We want to prove that system (2.5) is not observable from ω_x in time T with exponential cost d. Let g_n be as in (4.28). By (4.30) and (2.9), we have

$$\frac{e^{2dn} \int_0^T \int_{\omega_x} g_n(t,x)^2 \mathrm{d}x\mathrm{d}t}{\int_{-1}^1 g_n(T,x)^2 \mathrm{d}x} \underset{n \to +\infty}{\longrightarrow} 0. \tag{5.1}$$

which gives the conclusion.

Step 2: We prove Proposition 5.1 for $\gamma > 1$.

Step 2.1: We prove that $d^* \leqslant \pi \frac{a^{\gamma+1}}{\gamma+1}$. We want to prove that system (2.5) is observable from ω_x in time T with any exponential cost $d > \pi \frac{a^{\gamma+1}}{\gamma+1}$. We here apply the exact same strategy as in the case $\gamma = 1$, except that we do not use the dissipation of the 1D heat equation. Let L and δ satisfying the assumptions of Proposition 4.5. We apply Proposition 4.5 with the transmutation kernel k_T and $q(x) := (n\pi)^2 |x|^{2\gamma}$. The same computations as (4.18), (4.19) and (4.22) imply

$$\|g_n(T)\|^2_{L^2(-1,1)} \leqslant O_W(a,q,\delta,\tilde{\delta}) \int_0^T \int_{\omega_x} g_n(t,x)^2 \mathrm{d}x\mathrm{d}t,$$

where

$$O_W(a,q,\delta,\tilde{\delta}) \leqslant C \exp\left(2n\pi \left(\frac{(a+2\delta)^{\gamma+1}}{\gamma+1} + \tilde{\delta}(a+2\delta) \right) \right).$$

Step 2.2: We prove that $d^* \geqslant \pi \frac{a^{\gamma+1}}{\gamma+1}$. Let $d < \pi \frac{a^{\gamma+1}}{\gamma+1}$. We want to prove that system (2.5) is not observable from ω_x in time T with exponential cost d. Let $g_n(t,x) := v_{n,\gamma}(x) e^{-\lambda_{n,\gamma} t}$ where $v_{n,\gamma}$ and $\lambda_{n,\gamma}$ are as in Proposition 2.4. We have, for n large enough

$$v_n(x) \leqslant C_n e^{-n\pi \frac{x^{\gamma+1}}{\gamma+1}}, \quad \forall x \geqslant a, \tag{5.2}$$

where

$$C_n := \frac{2\lambda_n e^{\mu_n x_n^{\gamma+1}}}{(\gamma+1)\mu_n x_n^{\gamma-\frac{1}{2}}}, \quad x_n := \left(\frac{\lambda_n}{(n\pi)^2}\right)^{\frac{1}{2\gamma}},$$

(see, for instance, [6, Lemma 3]). Thus, for n large enough, we have

$$\int_a^b v_{n,\gamma}(x)^2 dx \leqslant C_n^2 e^{-2n\pi \frac{a^{\gamma+1}}{\gamma+1}}$$

Together with the estimates on $\lambda_{n,\gamma}$ in Proposition 2.4, this proves that (5.1) holds, which ends Step 2.2. $\square$

Remark 5.1. Agmon estimates state the exponential decrease as $e^{-d_{Ag}(x)/h}$, of the eigenfunctions of $-h^2\Delta + q(x)$, far from the minimal points of q; where $d_{Ag}(x)$ is the Agmon distance associated to potential q. Thus, the key estimate (5.2) of the above proof, is an Agmon estimate (take $h = 1/n\pi$ and $q(x) = |x|^{2\gamma}$).

6 Conclusion and open problems

In this article, we have studied the null controllability of degenerate parabolic equations, of Grushin type, on a rectangle domain, with strip shaped control supports.

We have proved that, when the control acts on a strip touching the degeneracy line $\{x = 0\}$, then, null controllability holds in arbitrary time $T > 0$ and for arbitrary degeneracy $\gamma > 0$. This result contrasts with the case of a strip that does not touch the degeneracy line (see Theorem 1.1 from [6]).

Then, we focused on control regions consisting in two symmetric strips.

In this setting, it was already known that a positive minimal time is required for null controllability when $\gamma = 1$ [6]. We here give the exact value of this minimal time, and its interpretation in terms of the Agmon distance between the control region and the degeneracy line.

It was also known that, in particular configurations ($\gamma > 1$ or $[\gamma = 1, T$ small$]$), null controllability does not hold. In these cases, we characterize the initial conditions that are null controllable: their regularity depends on the control region ω, the time T and the degeneracy parameter γ. This result may be understood as a geometric control condition.

Many questions are still open.

The results of the present article incites to conjecture that, for given degeneracy parameter γ, control region ω, and time T, the regularity of initial conditions that are null controllable depends on (γ, ω, T). This characterization is an open problem, when the control region does not consist in two symetric strips (even when it consists in one strip).

When the control region ω is not strip shaped, the existence of a critial parameter γ for which null controllability requires a positive minimal time is also an open problem.

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7 Appendix

Proposition 7.1. *Let $\epsilon \in (0,1)$. There exists a function $\beta \in C^1([-1,1]) \cap C^2([-1,0]) \cap C^2([0,1])$ such that*

$$\beta \geqslant 1 \text{ on } (-1,1), \tag{7.1}$$

$$\beta' < 0 \text{ on } [-1,0], \tag{7.2}$$

$$\beta'(-1) < 0 < \beta'(+1), \tag{7.3}$$

$$\beta'' < -1 \text{ on } [-1,0), \tag{7.4}$$

$$(|x|^{2\gamma}\beta')' \geqslant 0 \text{ on } [-1,0), \tag{7.5}$$

$$\beta'' \equiv 0 \text{ on } [0,\epsilon]. \tag{7.6}$$

Note that β'' has a discontinuity at $x = 0$.

Proof of Proposition 7.1. We define

$$\beta : x \in [-1,0) \mapsto -x^2 - \frac{2\gamma+1}{\gamma}x + 2 + \varepsilon\frac{2\gamma+1}{\gamma}.$$

By straightforward computations, this polynomial satisfies the assumptions on $[-1,0)$. From the C^1 regularity and (7.6) it comes that β must be defined on $[0,\varepsilon]$ by

$$\beta : x \in [0,\varepsilon] \mapsto -\frac{2\gamma+1}{\gamma}x + 2 + \varepsilon\frac{2\gamma+1}{\gamma}.$$

This construction ensures that $\beta \geqslant 2$ on $[0,\varepsilon]$. We end the construction on $[\varepsilon,1]$ with any function satisfying

$$\beta(\varepsilon) = 2, \quad \beta'(\varepsilon) = -\frac{2\gamma+1}{\gamma}, \quad \beta''(\varepsilon) = 0, \quad \beta \geqslant 1 \text{ on } [\varepsilon,1], \quad \beta'(1) > 0.$$

□

Proof of Proposition 3.2. All the computations of the proof will be made assuming, first, $g \in H^1((0,T), L^2(-1,1)) \cap L^2((0,T), H^2 \cap H^1_0(-1,1))$. Then, the conclusion of Proposition 3.2 will follow by a density argument.

We consider the weight function

$$\alpha(t,x) := \frac{M\beta(x)}{t(T-t)}, \quad (t,x) \in (0,T) \times \mathbb{R}, \tag{7.7}$$

where β is as in Proposition 7.1 and $M = M(T,\beta) > 0$ will be chosen later on. We also introduce the function

$$z(t,x) := g(t,x)e^{-\alpha(t,x)}, \tag{7.8}$$

that satisfies

$$e^{-\alpha}\mathcal{P}_{n,\gamma}g = P_1z + P_2z + P_3z, \tag{7.9}$$

where

$$\begin{aligned}
P_1z &:= -\frac{\partial^2 z}{\partial x^2} + (\alpha_t - \alpha_x^2)z + (n\pi)^2|x|^{2\gamma}z,\\
P_2z &:= \frac{\partial z}{\partial t} - 2\alpha_x\frac{\partial z}{\partial x},\\
P_3z &:= -\alpha_{xx}z.
\end{aligned} \tag{7.10}$$

We develop the classical proof (see [21]), taking the $L^2(Q)$-norm in the identity (7.9), then developing the double product, which leads to

$$\int_Q \left(P_1zP_2z - \frac{1}{2}|P_3z|^2\right)\mathrm{d}x\mathrm{d}t \leqslant \int_Q |e^{-\alpha}\mathcal{P}_{n,\gamma}g|^2\mathrm{d}x\mathrm{d}t, \tag{7.11}$$

where $Q := (0,T)\times(-1,1)$ and we compute precisely each term, paying attention to the behaviour of the different constants with respect to n and T.

Terms concerning $-\partial^2_{xx}z$**:** Integrating by parts, we get

$$-\int_Q \frac{\partial^2 z}{\partial x^2}\frac{\partial z}{\partial t}\mathrm{d}x\mathrm{d}t = \int_Q \frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial t\partial x}\mathrm{d}x\mathrm{d}t = \int_0^T \frac{1}{2}\frac{d}{dt}\int_{-1}^1 \left|\frac{\partial z}{\partial x}\right|^2\mathrm{d}x\mathrm{d}t = 0, \tag{7.12}$$

because $\partial_t z(t,\pm1) = 0$ and $z(0)\equiv z(T)\equiv 0$, which is a consequence of assumptions (7.8), (7.7) and (7.1). Moreover,

$$\begin{aligned}
\int_Q \frac{\partial^2 z}{\partial x^2}2\alpha_x\frac{\partial z}{\partial x}\mathrm{d}x\mathrm{d}t &= -\int_Q \left|\frac{\partial z}{\partial x}\right|^2\alpha_{xx}\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\left(\alpha_x(t,1)\left|\frac{\partial z}{\partial x}(t,1)\right|^2 - \alpha_x(t,-1)\left|\frac{\partial z}{\partial x}(t,-1)\right|^2\right)\mathrm{d}t.
\end{aligned} \tag{7.13}$$

Terms concerning $(\alpha_t - \alpha_x^2)z$**:** Again integrating by parts, we have

$$\int_Q (\alpha_t - \alpha_x^2)z\frac{\partial z}{\partial t}\mathrm{d}x\mathrm{d}t = -\frac{1}{2}\int_Q(\alpha_t-\alpha_x^2)_t|z|^2\mathrm{d}x\mathrm{d}t. \tag{7.14}$$

Indeed, the boundary terms at $t=0$ and $t=T$ vanish because, using (7.8), (7.7), (7.1), we get

$$|(\alpha_t - \alpha_x^2)|z|^2| \leqslant \frac{1}{[t(T-t)]^2}e^{\frac{-M}{t(T-t)}}|M(T-2t)\beta + (M\beta')^2|\,|g|^2$$

which tends to zero when $t\to 0$ and $t\to T$, for every $x\in[-1,1]$. Moreover,

$$-2\int_Q(\alpha_t-\alpha_x^2)z\alpha_x\frac{\partial z}{\partial x}\mathrm{d}x\mathrm{d}t = \int_Q[(\alpha_t-\alpha_x^2)\alpha_x]_x|z|^2\mathrm{d}x\mathrm{d}t, \tag{7.15}$$

thanks to an integration by parts in the space variable.

Terms concerning $(n\pi)^2|x|^{2\gamma}z$**:** First, since $z(0) \equiv z(T) \equiv 0$,

$$\int_Q (n\pi)^2 |x|^{2\gamma} z \frac{\partial z}{\partial t} \mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_0^T \frac{d}{dt}\int_{-1}^1 (n\pi)^2|x|^{2\gamma}|z|^2 \mathrm{d}x\mathrm{d}t = 0. \tag{7.16}$$

Furthermore, thanks to an integration by parts in the space variable,

$$-2\int_Q (n\pi)^2|x|^{2\gamma} z\alpha_x \frac{\partial z}{\partial x}\mathrm{d}x\mathrm{d}t = \int_Q [(n\pi)^2|x|^{2\gamma}\alpha_x]_x z^2 \mathrm{d}x\mathrm{d}t. \tag{7.17}$$

Combining (7.11), (7.12), (7.13), (7.14), (7.15), (7.16) and (7.17), we get

$$\begin{aligned}
&\int_Q |z|^2\Big\{-\frac{1}{2}(\alpha_t-\alpha_x^2)_t + [(\alpha_t-\alpha_x^2)\alpha_x]_x + (n\pi)^2[|x|^{2\gamma}\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2\Big\}\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\left(\alpha_x(t,1)\Big|\frac{\partial z}{\partial x}(t,1)\Big|^2 - \alpha_x(t,-1)\Big|\frac{\partial z}{\partial x}(t,-1)\Big|^2\right)\mathrm{d}t\\
&-\int_Q\Big|\frac{\partial z}{\partial x}\Big|^2\alpha_{xx}\mathrm{d}x\mathrm{d}t \leqslant \int_Q |e^{-\alpha}\mathcal{P}_{n,\gamma}g|^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{7.18}$$

In view of (7.3), we have $\alpha_x(t,1)\geqslant 0$ and $\alpha_x(t,-1)\leqslant 0$, thus (7.18) yields

$$\begin{aligned}
&\int_Q |z|^2\Big\{-\frac{1}{2}(\alpha_t-\alpha_x^2)_t + [(\alpha_t-\alpha_x^2)\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2 + n^2\pi^2[|x|^{2\gamma}\alpha_x]_x\Big\}\mathrm{d}x\mathrm{d}t\\
&-\int_Q\Big|\frac{\partial z}{\partial x}\Big|^2\alpha_{xx}\mathrm{d}x\mathrm{d}t \leqslant \int_Q |e^{-\alpha}\mathcal{P}_{n,\gamma}g|^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{7.19}$$

Now, in the left side of (7.19) we separate the terms on $(0,T)\times(-1,0)$ and those on $(0,T)\times(0,1)$. One has

$$\begin{aligned}
-\alpha_{xx}(t,x) &\geqslant \frac{C_1 M}{t(T-t)}, && \forall x\in(-1,0),\ t\in(0,T),\\
\alpha_{xx}(t,x) &= 0, && \forall x\in(0,\epsilon),\ t\in(0,T),\\
|\alpha_{xx}(t,x)| &\leqslant \frac{C_2 M}{t(T-t)}, && \forall x\in(\epsilon,1),\ t\in(0,T),
\end{aligned} \tag{7.20}$$

where $C_1 = C_1(\beta) := \inf\{-\beta''(x); x\in[-1,0)\}$ is positive thanks to the assumption (7.4) and $C_2 = C_2(\beta) := \sup\{|\beta''(x)|; x\in[\epsilon,1]\}$. Moreover,

$$\begin{aligned}
&-\frac{1}{2}(\alpha_t-\alpha_x^2)_t + [(\alpha_t-\alpha_x^2)\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2 = \frac{1}{(t(T-t))^3}\Big\{M\beta(3Tt-T^2-3t^2)\\
&+M^2\Big[(2t-T)(\beta''\beta+2\beta'^2) - \frac{t(T-t)\beta''^2}{2}\Big] - 3M^3\beta''\beta'^2\Big\}.
\end{aligned}$$

Hence, owing to (7.2) and (7.4), there exist $m_1 = m_1(\beta) > 0$, $C_3 = C_3(\beta) > 0$ and $C_4 = C_4(\beta) > 0$ such that, for every $M \geqslant M_1$ and $t\in(0,T)$,

$$\begin{aligned}
&-\frac{1}{2}(\alpha_t-\alpha_x^2)_t + [(\alpha_t-\alpha_x^2)\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2 \geqslant \frac{C_3M^3}{[t(T-t)]^3} \quad \forall x\in(-1,0),\\
&\Big|-\frac{1}{2}(\alpha_t-\alpha_x^2)_t + [(\alpha_t-\alpha_x^2)\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2\Big| \leqslant \frac{C_4M^3}{[t(T-t)]^3} \quad \forall x\in(0,1)
\end{aligned} \tag{7.21}$$

where

$$M_1 = M_1(T,\beta) := m_1(\beta)(T+T^2). \tag{7.22}$$

Note that, by assumption (7.5),

$$\begin{aligned} &(n\pi)^2[|x|^{2\gamma}\alpha_x]_x = (n\pi)^2\frac{M(|x|^{2\gamma}\beta')'}{t(T-t)} \geqslant 0, \quad \forall (t,x)\in(0,T)\times(-1,0), \\ &\Big|(n\pi)^2[|x|^{2\gamma}\alpha_x]_x\Big| \leqslant \frac{Mn^2C_4'}{t(T-t)}, \quad \forall (t,x)\in(0,T)\times(0,1), \end{aligned} \tag{7.23}$$

where $C_4' = C_4'(\beta,\gamma) > 0$. The key point of this proof is the first inequality in the above formula: outside the control support, the only term that depends on n is positive and can be neglected. In particular M does not need to depend on n, contrarily to what happens in reference [6]. Using (7.19), (7.20), (7.21) and (7.23) we get, for every $M \geqslant M_1$,

$$\begin{aligned} &\int_0^T \int_{(-1,0)} \left[\frac{C_1M}{t(T-t)}\Big|\frac{\partial z}{\partial x}\Big|^2 + \frac{C_3M^3}{(t(T-t))^3}|z|^2\right]\mathrm{d}x\mathrm{d}t \\ &\leqslant \int_Q |e^{-\alpha}\mathcal{P}_{n,\gamma}g|^2\mathrm{d}x\mathrm{d}t + \int_0^T\int_{(\epsilon,1)}\frac{C_2M}{t(T-t)}\Big|\frac{\partial z}{\partial x}\Big|^2\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T\int_{(0,1)}\left[\frac{C_4M^3}{(t(T-t))^3} + \frac{C_4'Mn^2}{t(T-t)}\right]|z|^2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{7.24}$$

For every $\delta > 0$,

$$\begin{aligned} &\frac{C_1M}{t(T-t)}\Big|\frac{\partial g}{\partial x} - \alpha_x g\Big|^2 + \frac{C_3M^3}{2(t(T-t))^3}|g|^2 \\ &\geqslant \left(1-\frac{1}{1+\delta}\right)\frac{C_1M}{t(T-t)}\Big|\frac{\partial g}{\partial x}\Big|^2 + \frac{M^3}{(t(T-t))^3}\left(\frac{C_3}{2} - \delta C_1(\beta')^2\right)|g|^2. \end{aligned} \tag{7.25}$$

Hence, choosing

$$\delta = \delta(\beta) := \frac{C_3}{4C_1\|\beta'\|_\infty^2},$$

from (7.24), (7.25) and (7.8) we deduce that

$$\begin{aligned} &\int_0^T\int_{(-1,0)}\frac{C_3M^3|g|^2}{4(t(T-t))^3}e^{-2\alpha}\mathrm{d}x\mathrm{d}t \\ &\leqslant \int_Q |e^{-\alpha}\mathcal{P}_{n,\gamma}g|^2\mathrm{d}x\mathrm{d}t + \int_0^T\int_{(\epsilon,1)}\frac{C_8M}{t(T-t)}\Big|\frac{\partial g}{\partial x}\Big|^2 e^{-2\alpha}\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T\int_{(0,1)}\left[\frac{C_9M^3}{(t(T-t))^3} + \frac{C_4'Mn^2}{t(T-t)}\right]|g|^2e^{-2\alpha}\mathrm{d}x\mathrm{d}t, \end{aligned}$$

where $C_8 = C_8(\beta) := 2C_2$ and $C_9 = C_9(\beta) := C_4 + 2C_2 \sup\{\beta'(x)^2 : x \in [0,1]\}$. So, adding the same quantity to both sides,

$$\begin{aligned}
&\int_Q \frac{C_3 M^3 |g|^2}{4(t(T-t))^3} e^{-2\alpha} \mathrm{d}x\mathrm{d}t \\
&\leqslant \int_Q |e^{-\alpha}\mathcal{P}_n g|^2 \mathrm{d}x\mathrm{d}t + \int_0^T \int_{(\epsilon,1)} \frac{C_8 M}{t(T-t)} \left|\frac{\partial g}{\partial x}\right|^2 e^{-2\alpha} \mathrm{d}x\mathrm{d}t \\
&+ \int_0^T \int_{(0,1)} \left[\frac{C_{11} M^3}{(t(T-t))^3} + \frac{C_4' M n^2}{t(T-t)}\right] |g|^2 e^{-2\alpha} \mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{7.26}$$

where $C_{11} = C_{11}(\beta) := C_9 + C_3/4$. Let us prove that the second term of the right hand side may be dominated by terms similar to the third one. We consider $\chi \in C^\infty(\mathbb{R}, \mathbb{R}_+)$ such that $0 \leqslant \chi \leqslant 1$ and

$$\chi \equiv 1 \text{ on } (\epsilon, 1), \tag{7.27}$$

$$\chi \equiv 0 \text{ on } (-1, 0). \tag{7.28}$$

We have

$$\int_Q (\mathcal{P}_n g) \frac{g\chi e^{-2\alpha}}{t(T-t)} \mathrm{d}x\mathrm{d}t = \int_0^T \int_{-1}^1 \left[\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} + (n\pi)^2 |x|^{2\gamma} g\right] \frac{g\chi e^{-2\alpha}}{t(T-t)} \mathrm{d}x\mathrm{d}t.$$

Integrating by parts with respect to time and space, we obtain

$$\int_Q \frac{1}{2} \frac{\partial(g^2)}{\partial t} \frac{\chi e^{-2\alpha}}{t(T-t)} \mathrm{d}x\mathrm{d}t = \int_Q \frac{1}{2} |g|^2 \chi \left(\frac{2\alpha_t}{t(T-t)} + \frac{T-2t}{(t(T-t))^2}\right) e^{-2\alpha} \mathrm{d}x\mathrm{d}t$$

and

$$\begin{aligned}
&- \int_Q \frac{\partial^2 g}{\partial x^2} \frac{g\chi e^{-2\alpha}}{t(T-t)} \mathrm{d}x\mathrm{d}t = \int_Q \frac{\chi e^{-2\alpha}}{t(T-t)} \left|\frac{\partial g}{\partial x}\right|^2 \mathrm{d}x\mathrm{d}t \\
&- \int_Q \frac{|g|^2 e^{-2\alpha}}{2t(T-t)} \left(\chi'' - 4\chi'\alpha_x + \chi(4\alpha_x^2 - 2\alpha_{xx})\right) \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_Q \mathcal{P}_n g \frac{g\chi e^{-2\alpha}}{t(T-t)} \mathrm{d}x\mathrm{d}t \geqslant \int_Q \frac{\chi e^{-2\alpha}}{t(T-t)} \left|\frac{\partial g}{\partial x}\right|^2 \mathrm{d}x\mathrm{d}t \\
&- \int_Q \frac{|g|^2 e^{-2\alpha}}{2t(T-t)} \left(\chi'' - 4\chi'\alpha_x + \chi\left(4\alpha_x^2 - 2\alpha_{xx} - 2\alpha_t - \frac{T-2t}{t(T-t)}\right)\right) \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^T\int_\epsilon^1 \frac{C_8 M}{t(T-t)}\Big|\frac{\partial g}{\partial x}\Big|^2 e^{-2\alpha}\mathrm{d}x\mathrm{d}t \leqslant \int_Q \frac{C_8 M\chi}{t(T-t)}\Big|\frac{\partial g}{\partial x}\Big|^2 e^{-2\alpha}\mathrm{d}x\mathrm{d}t\\
&\leqslant \int_Q \mathcal{P}_n g\frac{C_8 M g\chi e^{-2\alpha}}{t(T-t)}\mathrm{d}x\mathrm{d}t\\
&+\int_Q \frac{C_8 M|g|^2 e^{-2\alpha}}{2t(T-t)}\left(\chi''-4\chi'\alpha_x+\chi\left(4\alpha_x^2-2\alpha_{xx}-2\alpha_t-\frac{T-2t}{t(T-t)}\right)\right)\mathrm{d}x\mathrm{d}t\\
&\leqslant \int_Q |\mathcal{P}_n g|^2 e^{-2\alpha}\mathrm{d}x\mathrm{d}t+\int_0^T\int_{(0,1)}\frac{C_{12}M^3}{(t(T-t))^3}|g|^2 e^{-2\alpha}\mathrm{d}x\mathrm{d}t
\end{aligned}$$

for some constant $C_{12}=C_{12}(\beta,\chi)>0$ and when $M\geqslant M_2=M_2(T,\beta,\chi)$.

Combining (7.26) with the previous inequality, we get

$$\begin{aligned}
&\int_Q \frac{C_3 M^3|g|^2}{4(t(T-t))^3}e^{-2\alpha}\mathrm{d}x\mathrm{d}t\\
&\leqslant \int_Q 2|e^{-\alpha}\mathcal{P}_n g|^2\mathrm{d}x\mathrm{d}t+\int_0^T\int_{(0,1)}\left[\frac{C_{13}M^3}{(t(T-t))^3}+\frac{C_4' M n^2}{t(T-t)}\right]|g|^2 e^{-2\alpha}\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $C_{13}=C_{13}(\beta,\chi):=C_{11}+C_{12}$. Then, the global Carleman estimates (3.1) holds with

$$\mathcal{C}_1=\mathcal{C}_1(\beta):=\frac{C_3}{4\max\{2;C_{13};C_4'\}}.$$

□

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